Miscellaneous Calculus Problems

1. **(a)** Let $f(x)$ be a continuous real-valued function on $[0, 1]$. Prove that

$$\int_0^\pi x\,f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx .$$

(b) Evaluate $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.

(c) Let $g(x) = \dfrac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}}$. Show that on $[0, \pi]$, $g'(x) = -\dfrac{1}{2(1+\sin x)}$.

Hence, using (a) or otherwise, evaluate $\displaystyle\int_0^\pi g(x)\,dx$.

2. Prove that the *n*th derivative of the function y/x is

$$(-1)^n\frac{n!}{x^{n+1}}\left(y - x\frac{dy}{dx}+\frac{x^2}{2!}\frac{d^2y}{dx^2}-\frac{x^3}{3!}\frac{d^3y}{dx^3}+\ldots+(-1)^n\frac{x^n}{n!}\frac{d^ny}{dx^n}\right),$$ where y is any function of x .

Prove that $\displaystyle\frac{d^n}{dx^n}\left(\frac{e^{-x}}{x}\right) = \frac{(-1)^n}{x^{n+1}}\left(n! - \int_0^x t^n e^{-t}dt\right)$.

3. **(a)** Show that $\displaystyle\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

(b) Hence or otherwise show that $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx = -\frac{\pi}{2}\ln 2$.

4. **(a)** Prove that $\displaystyle a^{2n}-1 = (a^2-1)\prod_{r=1}^{n-1}\left(1-2a\cos\frac{r\pi}{n}+a^2\right)$.

(b) Hence or otherwise show that $\displaystyle\int_0^\pi \ln(1-2a\cos x+a^2)dx = \begin{cases}\pi\ln a^2 & , a^2>1\\ 0 & , a^2<1\end{cases}$

5. **(a)** Evaluate **(i)** $\displaystyle\int x\,f''(x)\,dx$ **(ii)** $\displaystyle\int f'(2x)\,dx$

(b) In each of the following cases, find $f(x)$:

(i) $f'(x^2) = \dfrac{1}{x}$ $(x > 0)$ **(ii)** $f'(\sin^2 x) = \cos^2 x$ **(iii)** $f'(\ln x) = \begin{cases}1 & ,0\le x\le 1\\ x & ,1<x<+\infty\end{cases}$ and $f(0) = 0$.

6. Estimate the values of the following integrals using the second mean-value theorem:

(a) $\displaystyle\int_4^9 \frac{dx}{(2+\sqrt{x})^2}$ **(b)** $\displaystyle\int_0^{\pi/4}\sqrt{1+\sin^4\theta}\,d\theta$ **(c)** $\displaystyle\int_{\pi/3}^{\pi}\frac{x\,dx}{1+\cos^2 x}$

(d) $\displaystyle\int_0^{\pi/2} x\sqrt{\sin x}\,dx$ **(e)** $\displaystyle\int_0^1 x^{\frac{1}{4}}e^{-x}dx$

7. For any real number $p \ge 1$ and $q \ge 1$, define $B(p , q) = \displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}dx$.

(a) Show that $B(p, q) = B(q, p)$.

(b) Show that

(i) if $p \geq 1$ and $q \geq 2$, then $B(p, q) = \dfrac{q-1}{p+q-1} B(p, q-1)$.

(ii) if $p \geq 2$ and $q \geq 1$, then $B(p, q) = \dfrac{p-1}{p+q-1} B(p-1, q)$.

(c) Show that if $p \geq 1$ and $q \geq 1$, and $p \geq m > 0$, then $\int_0^1 x^{p-1}(1-x^m)^{q-1}dx = \dfrac{1}{m} B\left(\dfrac{p}{m}, q\right)$.

8. Let a, b be real numbers such that $a < b$ and let m, n be positive integers.

(a) If for all real numbers x, u, $[(1+u)x - (au+b)]^{m+n} = \sum_{k=0}^{m+n} A_k(x)u^k$ …. (*)

Show that $A_k(x) = C_k^{m+n}(x-a)^k(x-b)^{m+n-k}$ for $k = 0, 1, \ldots, m+n$,
where C_k^{m+n} is the coefficient of t^k in the expansion of $(1+t)^{m+n}$.

(b) By integrating both sides of (*) with respect to x, or otherwise, calculate

$$\int_a^b (x-a)^m (x-b)^n \, dx$$

(c) By differentiating both sides of (*) with respect to x, or otherwise, find

$\dfrac{d^r}{dx^r}\left\{(x-a)^m(x-b)^n\right\}$ at $x = a$, where r is a positive integer.

9. The function $f(x)$ is periodic with period π and is integrable over the closed interval $[0, \pi]$.

Prove that $\int_{n\pi}^{(n+1)\pi} e^{-ax} f(x)\,dx = e^{-an\pi} \int_0^{\pi} e^{-ax} f(x)\,dx$ and deduce that

$$\int_0^{\infty} e^{-ax} f(x)\,dx = \frac{e^{a\pi}}{e^{a\pi}-1} \int_0^{\pi} e^{-ax} f(x)\,dx .$$

10. It is given that $f(x)$ is continuous and positive for all x, and y is defined by the equation

$$y = \frac{a + \int_0^x t\,f(t)\,dt}{b + \int_0^x f(t)\,dt}$$, where a and b are constants.

If $\dfrac{dy}{dx} = 0$ when $x = X$ and $y = Y$, show that $X = Y$.

11. Prove that $\int_a^b f_1(x) f_2(x)\,dx \leq \sqrt{\int_a^b [f_1(x)]^2 dx} \sqrt{\int_a^b [f_2(x)]^2 dx}$

(Hint : Consider $\int_a^b [f_1(x) - t f_2(x)]^2 dx \geq 0$)

12. Prove that $2e^{-\frac{1}{4}} \leq \int_0^2 e^{x^2-x}\,dx \leq 2e^2$.

13. The functions $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are defined by

$$F(\alpha) = \int_{-1}^{1} \frac{\sin\alpha}{x^2 + 2x\cos\alpha + 1}\,dx, \quad G(\alpha) = \int_{0}^{1} \frac{\sin\alpha}{x^2 + 2x\cos\alpha + 1}\,dx$$

(a) Show that $F(n\pi)$ and $G(n\pi)$ are zero for any integer n, that both F and G are periodic and that both F and G are odd functions.

(b) Show that $F(\alpha) = \tan^{-1}\left(\cot\frac{\alpha}{2}\right) + \tan^{-1}\left(\tan\frac{\alpha}{2}\right)$ and deduce the values of $F(\alpha)$ for $0 < \alpha < \pi$ and $-\pi < \alpha < 0$.

(c) Show that $G(\alpha) = \tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)$ and deduce the values of $G(\alpha)$ for $0 < \alpha < \pi$ and $-\pi < \alpha < 0$.

(d) Sketch on separate diagrams the graphs of $F(\alpha)$ and $G(\alpha)$ for $-2\pi \le \alpha \le 3\pi$.

14. Define $\ln x^n = \int_1^{x^n} \frac{dt}{t}$, for $x > 0$, use the substitution $t = u^n$ to prove that $\ln x^n = n \ln x$.

By considering the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = 1 + x$, or otherwise, show that, for $x > 0$, $\frac{x}{1+x} < \ln(1+x) < x$ and deduce that, as x decreases to zero, $\frac{1}{x}\ln(1+x)$ tends to 1.

A periodic function is defined by $\begin{cases} f(x) = \frac{1}{x}\ln(1+x) & ,\text{for } 0 < x \le 1 \\ f(x+1) = f(x) & ,\text{for all } x \end{cases}$.

Sketch the graph $y = f(x)$ for values of x from -2 to 2 .

15. **(a)** With the aid of a sketch of $y = \frac{1}{x}$, or otherwise , explain why $\frac{1}{r} < \int_{r-1}^{r} \frac{dx}{x} < \frac{1}{r-1}$ and

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} \quad \text{where} \quad r, n \in \mathbb{N} \setminus \{1\} .$$

(b) Deduce from **(a)** that $0 < \ln\frac{r}{r-1} - \frac{1}{r} < \frac{1}{r-1} - \frac{1}{r}$ and hence, or otherwise, show that

$$0 < \ln 2 - \sum_{r=n+1}^{2n} \frac{1}{r} < \frac{1}{2n}$$

(c) If $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = a_n$, show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = a_{2n} - a_n = \sum_{r=n+1}^{2n} \frac{1}{r}$

and deduce that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to $\ln 2$.

16. **(a)** Show that, for all $t > 0$, $x > 0$, the gradient of the graph of $y = \frac{1}{x^t}$ is negative and increases steadily as x increases. Use sketches showing appropriate points of this graph to illustrate the inequalities.

(1) $\int_r^{r+1} \frac{1}{x^t} dx < \frac{1}{2}\left[\frac{1}{r^t} + \frac{1}{(r+1)^t}\right]$ and **(2)** $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^t} dx > \frac{1}{r^t}$ which hold for $t > 0$, $r \geq 1$.

(b) For all $t > 0$, $G_n(t)$ is defined by $G_n(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \dots + \frac{1}{n^t} - \int_1^n \frac{1}{x^t} dx$.

By taking each of the inequalities **(1)**, **(2)** for suitable integer values of r and combining them, deduce that, for all $t > 0$, $\frac{1}{2} + \frac{1}{2n^t} < G_n(t) < \int_{1/2}^1 \frac{1}{x^t} dx + \int_n^{n+\frac{1}{2}} \frac{1}{x^t} dx$.

(c) Write down the value of $\int_{1/2}^1 \frac{1}{x^t} dx$ if $t \neq 1$ and explain why $\int_n^{n+\frac{1}{2}} \frac{1}{x^t} dx < \frac{1}{2n^t}$ if $t > 0$.

(d) Deduce that, if $t > 0$, $t \neq 1$, $\frac{1}{2} \leq \lim_{n\to\infty} G_n(t) \leq \left[\frac{1}{1-t}\left(1 - \frac{1}{2^{1-t}}\right)\right]$ and hence find the value of $\lim_{t\to 0^+}\left[\lim_{n\to\infty} G_n(t)\right]$. Using the definition of $G_n(t)$ or otherwise show that $\lim_{n\to\infty}\left[\lim_{t\to 0^+} G_n(t)\right]$ also exists, but has a different value from the previous double limit.

17. Suppose f is a twice differentiable function with $f''(x) < 0$ for all $x > 0$.
Show that if $0 < a < b$ then $f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$ for all $1 \geq \lambda \geq 0$.
By induction or otherwise deduce that if $a_1, a_2, \dots, a_n > 0$ then $f\left(\frac{1}{n}\sum_{i=1}^n a_i\right) \geq \frac{1}{n}\sum_{i=1}^n f(a_i)$.

Setting $f(x) = \ln x$ deduce that $\frac{1}{n}\sum_{i=1}^n a_i \geq \left(\prod_{i=1}^n a_i\right)^{1/n}$.

(Hint: Consider $g(\lambda) = f(\lambda a + (1-\lambda)b) - \lambda f(a) - (1-\lambda) f(b)$ as a function of λ.)

18. **(a)** Prove that if $f(x) = x - \frac{1}{3}\left(8\sin\frac{x}{2} - \sin x\right)$,

then $f'(x) = k\sin^4\frac{x}{4}$, where k is a constant.

(b) Hence show than if $x > 0$, $\frac{32}{15}\sin^5\frac{x}{4} < f(x) < \frac{32}{15}\left(\frac{x}{4}\right)^5$.

(c) Evaluate $f\left(\frac{\pi}{6}\right)$ in surd form, and use (b) to prove that $4(\sqrt{6} - \sqrt{2}) - 1$ is an approximation of π.

19. If n is a positive integer and x is a positive variable, show by differentiation that $\dfrac{(n+1+x)^{n+1}}{(n+x)^n}$ is an increasing function of x. Hence deduce that $\left(1+\dfrac{x}{n}\right)^n < \left(1+\dfrac{x}{n+1}\right)^{n+1}$.

If n is a positive integer and x is a positive variable smaller than n, determine which of the two expressions $\left(1-\dfrac{x}{n}\right)^n$ and $\left(1-\dfrac{x}{n+1}\right)^{n+1}$ is greater.

20. In the following we shall denote the n-th derivative $\dfrac{d^n f}{dx^n}$ of a function f by $f^{(n)}$ and define $f^{(0)} = f$.

(a) Prove that $(f \times g)^{(1)} = f^{(0)} \times g^{(1)} + f^{(1)} \times g^{(0)}$.

(b) Prove that $(f \times g)^{(n)} = \sum_{r=0}^{n} C_r^n f^{(r)} g^{(n-r)}$, for any positive integer n, where

$$C_r^n = \frac{n(n-1)...(n-r+1)}{r(r-1)...2.1}, \qquad C_0^n = 1 .$$

(c) Show that if $y = f(x)$ satisfies the equation $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2 y = 0$, where m is a positive integer, then $(x^2+1)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0$, for any positive integer n.

(d) Hence show that if $g(x) = f^{(m+1)}(x)$, then $\dfrac{g'(x)}{g(x)} = -\dfrac{(2m+1)x}{x^2+1}$ and consequently

$$g(x) = \frac{C}{(x^2+1)^{(2m+1)/2}}$$ for some constant C.

21. Let $I_n = \int \dfrac{d\theta}{(2+\cos\theta)^n}$, where n is any non-negative integer.

(a) Express $\int \dfrac{\cos\theta d\theta}{(2+\cos\theta)^n}$ in terms of In and In-1, for any n > 0.

(b) Express $\int \dfrac{\cos^2\theta d\theta}{(2+\cos\theta)^n}$ in terms of In, In-1, and In-2, for any n > 1.

(c) By using the results of (a) and (b), prove that, for any n > 1,

$$I_n = \frac{1}{3(n-1)}\left[-\frac{\sin\theta}{(2+\cos\theta)^{n-1}} - (n-2)I_{n-2} + 2(2n-3)I_{n-1}\right]$$

(d) Hence evaluate $\int_0^{2\pi/3} \dfrac{d\theta}{(2+\cos\theta)^2}$

22. If f satisfies the functional equation $f(xy) = f(x) + f(y)$ … (1)

for all x, y in its domain. Show that if f is a solution of (1) and if f is differentiable at each $x \neq 0$, then $f'(x) = \dfrac{f'(1)}{x}$, for each $x \neq 0$.

23. For any real numbers $\alpha < \beta$, we denote

$(\alpha, \beta) = \{ x : x \in \mathbb{R}$ and $\alpha < x < \beta \}$, $(\alpha, \infty) = \{ x : x \in \mathbb{R}$ and $x > \alpha \}$

(a) Define a function $h(x)$ on $(1, \infty)$ by $h(x) = \dfrac{x}{\ln x}$.

(i) Show that $h(x)$ has a local minimum at $x = e$,

(ii) Explain why $h(x) \geq e$ for all $x \in (1, \infty)$.

(b) Let $b > 1$. Show that the function $f(x) = \dfrac{x^b}{b^x}$, defined for $x > 1$ is increasing on $\left(1, \dfrac{b}{\ln b}\right)$

and decreasing on $\left(\dfrac{b}{\ln b}, \infty\right)$.

(c) Using **(a)** and **(b)**, deduce that if $1 < a < b < e$, then $a^b < b^a$.

24. **(a)** **(i)** For any $x \geq 0$, show that $(1+x)^n > \dfrac{n(n-1)}{2}x^2$ for any positive integer n.

By putting $x = \sqrt[n]{n} - 1$ in the above inequality, or otherwise, show that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

(ii) Evaluate the expression $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^3+n+1}{n^5+1}} = 1$.

(b) Find the absolute maximum of the function $f(x) = x^{1/x}$ on $[1, \infty)$. Hence, or otherwise, find the greatest value among the sequence $\{\sqrt[n]{n}\}$, $n = 1, 2, \ldots$ (It is known that $2 < e < 3$)

25. If $y = \sin^{-1}x + (\sin^{-1}x)^2$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx}$ is independent of x and deduce that, for $n > 1$,

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - x(2n+1)\frac{d^{n+1}y}{dx^{n+1}} - n^2\frac{d^ny}{dx^n} = 0 .$$

Show that the value $\dfrac{d^{2r+1}y}{dx^{2r+1}}$ when $x = 0$ is $\dfrac{1}{2^{2r}}\left\{\dfrac{(2r)!}{r!}\right\}^2$